

# 4. SEQUENCES AND SERIES



- Geometric progression (G.P.)
- $n^{\text{th}}$  term of a G.P.
- Sum of  $n$  terms of a G.P.
- Sum of infinite terms of a G.P.
- Sigma notation.

## 4.1 SEQUENCE :

A set of numbers, where the numbers are arranged in a definite order, is called a sequence.

Natural numbers is an example of a sequence.

In general, a sequence is written as  $\{t_n\}$ .

**Finite sequence** – A sequence containing finite number of terms is called a finite sequence.

**Infinite sequence** – A sequence is said to be infinite if it is not a finite sequence.

In this case for every positive integer  $n$ , there is a unique  $t_n$  in the sequence.

Sequences that follow specific rule are called progressions.

In the previous class, we have studied Arithmetic Progression (A.P.).

In a sequence if the difference between any term and its preceding term ( $t_{n+1} - t_n$ ) is constant, for all  $n \in \mathbb{N}$  then the sequence is called an Arithmetic Progression (A.P.)

Consider the following sequences.

- 1) 2, 5, 8, 11, 14, .....
- 2) 4, 10, 16, 22, 28, .....
- 3) 4, 16, 64, 256, .....

4)  $\frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \dots$

The sequences 1) and 2), are in A.P. because  $t_{n+1} - t_n$  is constant.

But in sequences 3) and 4), the difference is not constant. In these sequences, the ratio of any term to its preceding term that is  $\frac{t_{n+1}}{t_n}$  is constant [ $n \in \mathbb{N}$ ].

Such a sequence is called a 'GEOMETRIC PROGRESSION' (G. P.).



## 4.2 GEOMETRIC PROGRESSION (G.P.)

**Definition :** A Sequence  $\{t_n\}$  is said to be a

Geometric Progression if  $\frac{t_{n+1}}{t_n} = \text{constant}$ .

$\frac{t_{n+1}}{t_n}$  is called the common ratio of the G.P. and

it is denoted by  $r$  ( $r \neq 0$ ), for all  $n \in \mathbb{N}$ .

It is a convention to denote the first term of the geometric progression by  $a$  ( $a \neq 0$ ).

The terms of a geometric progression with first term 'a' and common ratio 'r' are as follows.

$$a, ar, ar^2, ar^3, ar^4, \dots$$

Let's see some examples of G.P.

- (i) 2, 8, 32, 128, 512, ..... is a G.P. with  $a = 2$  and  $r = 4$ .
- (ii) 25, 5, 1,  $\frac{1}{5}$ , ..... is a G.P. with  $a = 25$  and  $r = \frac{1}{5}$ .

### 4.3 General term or the n<sup>TH</sup> term of a G.P.

If a and r are the first term and common ratio of G.P. respectively, then its general term is given by  $t_n = ar^{n-1}$ .

Let's find n<sup>th</sup> term of the following G.P.

i) 2, 8, 32, 128, 512, .....

Here a=2, r = 4

$$t_n = ar^{n-1} = 2 (4)^{n-1}$$

ii) 25, 5, 1,  $\frac{1}{5}$ , .....

Here a = 25, r =  $\frac{1}{5}$

$$t_n = ar^{n-1} = 25 \left(\frac{1}{5}\right)^{n-1}$$

#### 4.3.1 Properties of Geometric Progression.

- i) Reciprocals of terms of a G.P. are also in G.P.
- ii) If each term of a G.P. is multiplied or divided by a non zero constant, then the resulting sequence is also a G.P.
- iii) If each term of a G.P. is raised to the same power, the resulting sequence is also a G.P.

### SOLVED EXAMPLES

**Ex.1)** For the following G.P.s find the n<sup>th</sup> term  
3, -6, 12, -24, .....

**Solution:**

Here a=3, r = -2

$$\therefore t_n = ar^{n-1} = 3 (-2)^{n-1}$$

**Ex 2)** Verify whether  $1, \frac{-3}{2}, \frac{9}{4}, \frac{-27}{8}, \dots$

is a G.P. If it is a G.P., find its ninth term.

**Solution :** Here  $t_1 = 1, t_2 = \frac{-3}{2}, t_3 = \frac{9}{4}$ ,

$$\text{Consider } \frac{t_2}{t_1} = \frac{\frac{-3}{2}}{1} = \frac{-3}{2}$$

$$\frac{t_3}{t_2} = \frac{\frac{9}{4}}{\frac{-3}{2}} = \frac{-3}{2}$$

$$\frac{t_4}{t_3} = \frac{\frac{-27}{8}}{\frac{9}{4}} = \frac{-3}{2}$$

Here the ratio of any term to its previous term is constant hence the given sequence is a G.P.

$$\text{Now } t_9 = ar^{9-1} = ar^8 = 1 \left(\frac{-3}{2}\right)^8 = \frac{6561}{256}$$

**Ex 3)** For a G.P. if a = 3 and  $t_7 = 192$ , find r and

$$t_{11}.$$

**Solution :** Given a = 3,  $t_7 = ar^6 = 192$

$$\therefore 3 (r)^6 = 192, r^6 = \frac{192}{3} = 64$$

$$\therefore r^6 = 2^6,$$

$$\therefore r = \pm 2.$$

$$t_{11} = ar^{10} = 3 (\pm 2)^{10} = 3(1024) = 3072.$$

**Ex 4)** For a G.P.  $t_3 = 486, t_6 = 18$ , find  $t_{10}$

**Solution :** We know that  $t_n = ar^{n-1}$

$$t_3 = ar^2 = 486 \text{ ----- (1)}$$

$$t_6 = ar^5 = 18 \text{ ----- (2)}$$

dividing equation (2) by (1) we get,

$$\frac{t_6}{t_3} = \frac{ar^5}{ar^2} = \frac{18}{486} = \frac{2}{54} = \frac{1}{27}$$

$$r^3 = \left(\frac{1}{3}\right)^3, r = \frac{1}{3}.$$

Now from (1)  $ar^2 = 486$

$$a \left( \frac{1}{3} \right)^2 = 486.$$

$$a \left( \frac{1}{9} \right) = 486,$$

$$a = 486 \times 9$$

$$a = 4374.$$

$$\text{Now } t_{10} = ar^9 = 4374 \left( \frac{1}{3} \right)^9$$

$$= \frac{243 \times 2 \times 9}{3^9} = \frac{3^5 \times 2 \times 3^2}{3^5 \times 3^2 \times 3^2}$$

$$\therefore t_{10} = \frac{2}{9}$$

**Ex 5)** If for a sequence  $\{t_n\}$ ,  $t_n = \frac{5^{n-2}}{4^{n-3}}$ , show that the sequence is a G.P.

Find its first term and the common ratio.

**Solution:**  $t_n = \frac{5^{n-2}}{4^{n-3}}$

$$t_{n+1} = \frac{5^{n-1}}{4^{n-2}}$$

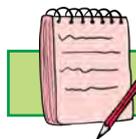
Consider  $\frac{t_{n+1}}{t_n} = \frac{5^{n-1}}{4^{n-2}} \times \frac{4^{n-3}}{5^{n-2}}$

$$= \frac{5^{n-1}}{4^{n-2}} \times \frac{4^{n-3}}{5^{n-2}} = \frac{5}{4} = \text{constant},$$

$$\forall n \in \mathbb{N}.$$

The given sequence is a G.P. with  $r = \frac{5}{4}$  and

$$t_1 = a = \frac{5^{1-2}}{4^{1-3}} = \frac{5^{-1}}{4^{-2}} = \frac{16}{5}.$$



**Let's Note.**

i) To find 3 numbers in G.P., it is convenient to take the numbers as

$$\frac{a}{r}, a, ar$$

ii) 4 numbers in a G.P. as  $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$ , (here the ratio is  $r^2$ )

iii) 5 numbers in a G.P. as  $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

**Ex 6)** Find three numbers in G.P. such that their sum is 42 and their product is 1728.

**Solution:** Let the three numbers be  $\frac{a}{r}, a, ar$ .

According to first condition their sum is 42

$$\therefore \frac{a}{r} + a + ar = 42$$

$$a \left( \frac{1}{r} + 1 + r \right) = 42$$

$$\therefore \frac{1}{r} + 1 + r = \frac{42}{a}$$

$$\frac{1}{r} + r = \frac{42}{a} - 1 \dots\dots\dots(1)$$

From the second condition their product is 1728

$$\frac{a}{r} \cdot a \cdot ar = 1728$$

$$\therefore a^3 = 1728 = (12)^3$$

$$\therefore a = 12.$$

substitute  $a = 12$  in equation (1), we get

$$\frac{1}{r} + r = \frac{42}{12} - 1$$

$$\frac{1}{r} + r = \frac{42-12}{12}$$

$$\frac{1}{r} + r = \frac{30}{12}$$

$$\frac{1+r^2}{r} = \frac{5}{2}$$

$$\therefore 2 + 2r^2 = 5r$$

$$\therefore 2r^2 - 5r + 2 = 0$$

$$\therefore (2r-1)(r-2) = 0$$

$$\therefore 2r=1 \text{ or } r=2$$

$$\therefore r = \frac{1}{2} \text{ or } r=2$$

Now if  $a=12$ , and  $r = \frac{1}{2}$  then the required numbers are 24, 12, 6.

If  $a = 12$ , and  $r = 2$  then the required numbers are 6,12,24.

$\therefore$  24,12,6 or 6,12,24 are the three required numbers in G.P.

**Ex 7)** In a G.P. ,if the third term is  $\frac{1}{5}$  and sixth term is  $\frac{1}{625}$ , find its  $n^{\text{th}}$  term .

**Solution :** Here  $t_3 = \frac{1}{5}$ ,  $t_6 = \frac{1}{625}$

$$t_3 = ar^2 = \frac{1}{5} \quad \dots\dots\dots(1) ,$$

$$t_6 = ar^5 = \frac{1}{625} \quad \dots\dots\dots(2)$$

Divide equation (2) by equation (1)

we get,

$$r^3 = \frac{1}{125} = \frac{1}{5^3}$$

$$\therefore r = \frac{1}{5} . \quad \text{Substitute } r = \frac{1}{5} \text{ in equation (1)}$$

we get

$$a \left( \frac{1}{5} \right)^2 = \frac{1}{5}$$

$$\therefore a = 5.$$

$$t_n = ar^{n-1} = 5 \left( \frac{1}{5} \right)^{n-1} = 5 \times 5^{1-n} = 5^{2-n}.$$

**Ex 8)** Find four numbers in G. P. such that their product is 64 and sum of the second and third number is 6.

**Solution :** Let the four numbers be  $\frac{a}{r^3}$ ,  $\frac{a}{r}$ ,  $ar$ ,  $ar^3$  (common ratio is  $r^2$ )

According to the first condition

$$\frac{a}{r^3} \times \frac{a}{r} \times ar \times ar^3 = 64$$

$$\therefore a^4 = 64$$

$$\therefore a = 2\sqrt{2}.$$

Now using second condition  $\frac{a}{r} + ar = 6$

$$\frac{2\sqrt{2}}{r} + 2\sqrt{2}r = 6. \text{ dividing by 2 we get ,}$$

$$\frac{\sqrt{2}}{r} + \sqrt{2} r = 3 \text{ now multiplying by } r \text{ we get}$$

$$\sqrt{2} + \sqrt{2} r^2 - 3r = 0$$

$$\sqrt{2} r^2 - 3r + \sqrt{2} = 0 ,$$

$$\sqrt{2} r^2 - 2r - r + \sqrt{2} = 0,$$

$$\sqrt{2} r (r - \sqrt{2}) - 1 (r - \sqrt{2}) = 0.$$

$$r = \sqrt{2} \text{ or } r = \frac{1}{\sqrt{2}} .$$

If  $a = 2\sqrt{2}$ , and  $r = \sqrt{2}$  then 1 ,2 , 4, 8 are the four required numbers

If  $a = 2\sqrt{2}$ , and  $r = \frac{1}{\sqrt{2}}$  then 8 ,4, 2, 1 are the

four required numbers in G.P.

**Ex 9)** If p,q,r,s are in G.P. then show that

$$(q-r)^2 + (r-p)^2 + (s-q)^2 = (p-s)^2$$

**Solution :** As p,q,r,s are in G.P.  $\frac{q}{p} = \frac{r}{q} = \frac{s}{r} = k$  (say)

$$\therefore q^2 = pr , r^2 = qs , qr = ps$$

consider L.H.S.

$$= (q-r)^2 + (r-p)^2 + (s-q)^2$$

$$= q^2 - 2qr + r^2 + r^2 - 2rp + p^2 + s^2 - 2sq + q^2$$

$$= pr - 2qr + qs + qs - 2rp + p^2 + s^2 - 2sq + pr$$

$$= -2qr + p^2 + s^2 = -2ps + p^2 + s^2 \quad (qr = ps)$$

$$= (p-s)^2 = \text{R.H.S.}$$

**Ex 10)** Shraddha deposited Rs. 8000 in a bank which pays annual interest rate of 8%.She kept it with the bank for 10 years with compound interest. Find the total amount she will receive after 10 years. [ given  $(1.08)^{10} = 2.1589$  ]

**Solution:**

The Amount deposited in a bank is Rs 8000 with 8% compound interest.

Each year, the ratio of the amount to the principal of that year is constant =  $\frac{108}{100}$

Hence we get a G.P. of successive amounts.

For P = 8000,

the amount after 1 year is  $8000 \times \frac{108}{100}$

the amount after 2 years is  $8000 \times \frac{108}{100} \times \frac{108}{100}$

the amount after 3 years is  $8000 \times \frac{108}{100} \times \frac{108}{100} \times \frac{108}{100}$ .

Therefore after 10 years the amount is

$$8000 \left(\frac{108}{100}\right)^{10} = 8000 (1.08)^{10} \\ = 8000 \times 2.1589 = 17271$$

Thus Shraddha will get Rs 17271 after 10 years.

**Ex 11)** The number of bacteria in a culture doubles every hour. If there were 50 bacteria originally in the culture, how many bacteria will be there after 5 hours ?

**Solution :** Given that the number of bacteria doubles every hour .

The ratio of bacteria after 1 hour to that at the beginning is 2

after 1 hour =  $50 \times 2$

after 2 hour =  $50 \times 2^2$

after 3 hour =  $50 \times 2^3$

Hence it is a G.P. with  $a=50$  ,  $r = 2$  .

To find the number of bacteria present after 5 hours, that is to find  $t_5$  .

$$t_5 = ar^5 = 50 (2)^5 = 50 (32) = 1600$$

**ACTIVITIES**

**Activity 4.1:**

Verify whether  $1, \frac{-4}{3}, \frac{16}{9}, \frac{-64}{27}, \dots$  is a G.P.

If it is a G.P. Find its ninth term.

**Solution :** Herer  $t_1 = 1, t_2 = \square, t_3 = \frac{16}{9}$ ,

Consider  $\frac{t_2}{t_1} = \square = \frac{-4}{3}$

$$\frac{t_4}{t_3} = \frac{\frac{-64}{27}}{\frac{16}{9}} = \square$$

Here the ratio is constant. Hence the given sequence is a G.P.

$$\text{Now } t_9 = \square = ar^8 = 1 \left(\frac{-4}{3}\right)^8 = \square$$

**Activity 4.2:**

For a G.P.  $a=3, r = 2, S_n = 765$ , find n.

**Solution :**  $S_n = 765 = \square (2^n - 1)$ ,

$$\frac{765}{3} = 255 = 2^n - 1,$$

$$2^n = \square = 2^8, n = \square$$

**Activity 4.3:**

For a G.P. if  $t_6 = 486, t_3 = 18$ , find  $t_9$

**Solution :** We know that, for a G.P.  $t_n = \square$

$$\therefore t_3 = \square = 18 \dots \text{(I)}$$

$$t_6 = \square = 486 \dots \text{(II)}$$

$$\therefore \frac{t_6}{t_3} = \frac{\square}{\square} = \frac{486}{18}$$

$$\therefore r^{\square} = \square$$



$$\therefore r = \square$$

Now from (I),  $t_3 = ar^2 = 18$

$$\therefore a = \square$$

$$\therefore t_9 = ar^8 = \square$$

**Activity 4.4:**

If  $a, b, c, d$  are in G.P. then show that  $(a - b)$ ,  $(b - c)$  and  $(c - d)$  are also in G.P.

**Solution :**  $a, b, c, d$  are in G.P.

$$\therefore b^2 = \square$$

$$\square = bd$$

$$ad = \square$$

To prove that  $(a - b)$ ,  $(b - c)$ ,  $(c - d)$  are in G.P.

i.e. to prove that  $(b - c)^2 = (a - b)(c - d)$

$$\begin{aligned} \therefore \text{RHS} &= (a - b)(c - d) \\ &= ac - \square - bc + \square \\ &= b^2 - bc - bc + \square \\ &= b^2 - 2bc + c^2 \\ &= \square \\ &= \text{LHS} \end{aligned}$$

**Activity 4.5:**

For a sequence,  $S_n = 7(4^n - 1)$ , find  $t_n$  and show that the sequence is a G.P.

**Solution :**

$$\begin{aligned} S_n &= 7(4^n - 1) \\ S_{(n-1)} &= 7 \square \\ t_n &= \square \\ &= 7(4^n - 1) - 7(\square) \\ &= 7[4^n - 1 - 4^{n-1} + 1] \\ &= 7 \square \end{aligned}$$

**Activity 4.6:**

10 people visited an exhibition on the first day. The number of visitors was doubled on the next day and so on. Find i) number of visitors on 9<sup>th</sup> day. ii) Total number of visitors after 12 days.

**Solution :** On 1<sup>st</sup> day number of visitors was  $\square$   
Number of visitors doubles on next day.

$$\therefore \text{On 2<sup>nd</sup> day number of visitors} = \square$$

$$\therefore \text{On 3<sup>rd</sup> day number of visitors} = \square$$

and so on

$$\therefore \text{Number of visitors are } 10, 20, 40, 80, \dots$$

These number forms a G.P. with  $a = \square$

$$r = \square$$

$$\begin{aligned} \therefore \text{No. of visitors on 9<sup>th</sup> day i.e. } t_9 &= ar^{9-1} \\ &= \square 2^{\square} \\ &= 10 \times 2^{\square} \\ &= 10 \times \square = \end{aligned}$$

Total number of visitors after 12 days

$$\begin{aligned} &= S_{12} = a \square \\ &= 10 \left[ \frac{1 - 2^{12}}{1 - 2} \right] = \frac{10 \times (1 - 4096)}{\square} \\ &= 10 \times 4095 \\ &= \square \end{aligned}$$

**Activity 4.7:**

Complete the following activity to find sum to  $n$  terms of  $7+77+777+7777+\dots$

$$\begin{aligned} \text{Let } S_n &= 7+77+777+7777+\dots \text{ upto } n \text{ terms} \\ &= 7 \times (\square + \square + \square + \square \dots \text{ upto } n \text{ terms}) \\ &= \frac{7}{9} (\square + \square + \square + \square \dots \text{ upto } n \text{ terms}) \\ &= \frac{7}{9} [(10-1) + (-) + (-) + (-) + (-) \dots \text{ upto } n \text{ terms}] \end{aligned}$$

$$= \frac{7}{9} [(10+10^2+\dots\dots\dots+ \text{upto } n \text{ terms}) \\ - (1+1+\dots\dots\dots \text{upto } n \text{ terms})]$$

$$= \frac{7}{9} [\square(\square) - \square] \dots\dots \text{using } \left( \frac{a(r^n - 1)}{r - 1} \right)$$

$$S_n = \frac{7}{9} \left( \frac{\square}{\square} (\square) - n \right) = \square$$

**Activity 4.8:**

An empty bus arrived at a bus stand. In the first minute two persons boarded the bus. In the second minute 4 persons, in the third minute 8 persons boarded the bus and so on. The bus was full to its seating capacity in 5 minutes. What was the number of seats in the bus?

**Solution :** In the first minute, 2 persons board. In the second minute, 4 persons board. and so on .....

Hence it is a

with a = , r =

The bus was full in 5 minutes

$$S_5 = a \left( \frac{r^5 - 1}{r - 1} \right) = 2 \left( \frac{\square - 1}{\square - 1} \right)$$

$$= 2 \left( \frac{\square}{\square} \right) = \square$$

The number of seats in the bus =



**Let's remember!**

For a G.P.  $\{t_n\}$ ,

- 1)  $\frac{t_{n+1}}{t_n} = \text{constant}, \forall n \in \mathbb{N}$
- 2)  $t_n = a r^{n-1}, a \neq 0, r \neq 0, \forall n \in \mathbb{N}$

- 3) 3 successive terms in G. P. are written as  $\frac{a}{r}, a, ar$ .
- 4) 4 successive terms in G. P. are written as  $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$ . (ratio  $r^2$ )
- 5) 5 successive terms in G. P. are written as  $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

**EXERCISE 4.1**

- 1) Verify whether the following sequences are G.P. If so, find  $t_n$ 
  - i) 2,6,18,54,.....
  - ii) 1,-5,25,-125,.....
  - iii)  $\sqrt{5}, \frac{1}{\sqrt{5}}, \frac{1}{5\sqrt{5}}, \frac{1}{25\sqrt{5}}, \dots\dots\dots$
  - iv) 3, 4, 5, 6,.....
  - v) 7, 14, 21, 28,.....
- 2) For the G.P.
  - i) if  $r = \frac{1}{3}, a = 9$ ; find  $t_7$
  - ii) if  $a = \frac{7}{243}, r = \frac{1}{3}$  find  $t_3$
  - iii) if  $a = 7, r = -3$  find  $t_6$
  - iv) if  $a = \frac{2}{3}, t_6 = 162$ , find  $r$
- 3) Which term of the G.P. 5,25,125,625,.....is  $5^{10}$ ?
- 4) For what values of  $x$ .
 
$$\frac{4}{3}, x, \frac{4}{27}$$
 are in G.P. ?
  - 5) If for a sequence,  $t_n = \frac{5^{n-3}}{2^{n-3}}$ , show that the sequence is a G.P. Find its first term and the common ratio.

- 6) Find three numbers in G.P. such that their sum is 21 and sum of their squares is 189.
- 7) Find four numbers in G.P. such that sum of the middle two numbers is  $10/3$  and their product is 1.
- 8) Find five numbers in G. P. such that their product is 1024 and fifth term is square of the third term.
- 9) The fifth term of a G.P. is  $x$ , eighth term of the G.P. is  $y$  and eleventh term of the G.P. is  $z$ . Verify whether  $y^2 = xz$ .
- 10) If  $p, q, r, s$  are in G.P. show that  $p+q, q+r, r+s$  are also in G.P.



### Let's learn.

#### 4.4 Sum of the first n terms of a G.P.

If  $\{t_n\}$  is a geometric progression with first term  $a$  and common ratio  $r$ ; where  $a \neq 0, r \neq 0$ ; then the sum of its first  $n$  terms is given by

$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1} =$$

$$a \left( \frac{1-r^n}{1-r} \right), r \neq 1$$

**Proof :** Consider

$$S = 1 + r + r^2 + r^3 + \dots + r^{n-1} \quad \dots \dots \dots (i)$$

Multiplying both sides by  $r$  we get

$$rS = r + r^2 + r^3 + \dots + r^n \quad \dots \dots \dots (ii)$$

Subtract (ii) from (i) we get  $S - rS = 1 - r^n$

$$\therefore S(1-r) = 1 - r^n$$

$$\therefore S = \left( \frac{1-r^n}{1-r} \right), \dots \dots \dots (iii), r \neq 1.$$

Multiplying both sides of equation (i) by  $a$  we get ,

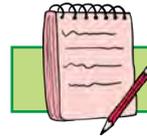
$$aS = a + ar + ar^2 + \dots + ar^{n-1} = S_n$$

$$a \left( \frac{1-r^n}{1-r} \right) = S_n \text{ \{from (i) and (iii)\}}$$

$$S_n = a \left( \frac{1-r^n}{1-r} \right), r \neq 1.$$

If we Subtract (i) from (ii) we get,

$$S_n = a \left( \frac{r^n - 1}{r - 1} \right).$$



### Let's Note.

1.  $S_n = a \cdot S$
2. If  $r = 1$ , then  $S_n = na$

#### Solved Examples

**Ex 1)** If  $a = 1, r = 2$  find  $S_n$  for the G.P.

**Solution :**  $a = 1, r = 2$

$$S_n = a \left( \frac{1-r^n}{1-r} \right) = 1 \left( \frac{1-2^n}{1-2} \right) = 2^n - 1.$$

**Ex 2)** For a G.P. 0.02, 0.04, 0.08, 0.16, ..., find  $S_n$ .

**Solution :** Here  $a = 0.02, r = 2$

$$S_n = a \left( \frac{1-r^n}{1-r} \right) = 0.02 \left( \frac{1-2^n}{1-2} \right) \\ = 0.02 \cdot (2^n - 1)$$

**Ex 3)** For the G.P. 3, -3, 3, -3, ..., Find  $S_n$ .

**Solution :**

If  $n$  is even ,  $n = 2k$

$$S_{2k} = (3-3) + (3-3) + (3-3) + \dots + (3-3) = 0.$$

If  $n$  is odd ,  $n = 2k + 1$

$$S_{2k+1} = S_{2k} + t_{2k+1} = 0 + 3 = 3.$$

**Ex 4)** For a G.P. if  $a=6, r=2$ , find  $S_{10}$ .

**Solution:**  $S_n = a \left( \frac{1-r^n}{1-r} \right)$ ,

$$S_{10} = 6 \left( \frac{1-2^{10}}{1-2} \right) = 6 \left( \frac{1-1024}{-1} \right)$$

$$= 6 \left( \frac{-1023}{-1} \right) = 6(1023) = 6138.$$

**Ex 5)** If for a G.P.  $r=2$ ,  $S_{10}=1023$ , find  $a$ .

**Solution :**  $S_{10} = a \left( \frac{1-2^{10}}{1-2} \right)$

$$\therefore 1023 = a(1023)$$

$$\therefore a = 1.$$

**Ex 6)** For a G.P.  $a = 5$ ,  $r = 2$ ,  $S_n = 5115$ , find  $n$ .

**Solution :**  $S_n = 5115 = 5 \left( \frac{2^n - 1}{2 - 1} \right) = 5(2^n - 1)$

$$\therefore \frac{5115}{5} = 1023 = 2^n - 1$$

$$2^n = 1024 = 2^{10}$$

$$\therefore n = 10$$

**Ex 7)** If for a G.P.  $S_3 = 16$ ,  $S_6 = 144$ , find the first term and the common ratio of the G.P.

**Solution :** Given

$$S_3 = a \left( \frac{1-r^3}{1-r} \right) = 16 \dots\dots\dots (1)$$

$$S_6 = a \left( \frac{1-r^6}{1-r} \right) = 144 \dots\dots\dots (2)$$

Dividing (2) by (1) we get ,

$$\frac{S_6}{S_3} = \frac{r^6 - 1}{r^3 - 1} = \frac{144}{16} = 9$$

$$\frac{(r^3 - 1)(r^3 + 1)}{(r^3 - 1)} = 9,$$

$$(r^3 + 1) = 9,$$

$$r^3 = 8 = 2^3,$$

$$r = 2.$$

Substitute  $r = 2$  in (1) We get

$$a \left( \frac{1-2^3}{1-2} \right) = 16,$$

$$a \left( \frac{1-8}{1-2} \right) = 16,$$

$$a(7) = 16,$$

$$a = \frac{16}{7}$$

**Ex 8)** Find the sum

$$9+99+999+9999+\dots\dots\dots \text{upto } n \text{ terms.}$$

**Solution :** Let  $S_n = 9+99+999+9999+\dots\dots\dots$   
 $\dots\dots\dots$  upto  $n$  terms.

$$S_n = (10-1) + (100-1) + (1000-1) \dots \text{ to } n \text{ brackets.}$$

$$= (10+100+1000+ \dots\dots \text{ upto } n \text{ terms})$$

$$- (1+1+1 \dots\dots \text{ upto } n \text{ terms})$$

Terms in first bracket are in G.P. with  $a = 10$ ,  
 $r = 10$  and terms in second bracket are in G.P. with  
 $a = r = 1$

$$\therefore S_n = 10 \left( \frac{10^n - 1}{10 - 1} \right) - n$$

$$= \frac{10}{9} (10^n - 1) - n.$$

**Ex 9)** Find the sum  $5+55+555+5555+\dots\dots\dots$  upto  $n$  terms.

**Solution:** Let  $S_n$

$$= 5+55+555+5555+\dots\dots\dots \text{ upto } n \text{ terms.}$$

$$= 5(1+11+111+ \dots\dots\dots \text{ upto } n \text{ terms})$$

$$= \frac{5}{9} (9+99+999+ \dots\dots\dots \text{ upto } n \text{ terms})$$

$$= \frac{5}{9} [(10-1) + (100-1) + (1000-1) + \dots\dots\dots$$

$$\text{to } n \text{ brackets}]$$

$$= \frac{5}{9} [(10 + 100 + 1000 + \dots\dots\dots \text{ upto } n \text{ terms})$$

$$\begin{aligned}
 & - (1+1+1+ \dots \text{ upto } n \text{ terms})] \\
 & = \frac{5}{9} \left[ 10 \left( \frac{10^n - 1}{10 - 1} \right) - n \right] \\
 & = \frac{5}{9} \left[ \frac{10}{9} (10^n - 1) - n \right]
 \end{aligned}$$

**Ex 10)** Find the sum to  $n$  terms

$$0.3+0.03+0.003+\dots \text{ upto } n \text{ terms}$$

**Solution :** Let  $S_n$

$$\begin{aligned}
 & = 0.3+0.03+0.003+ \dots \text{ upto } n \text{ terms} \\
 & = 3 [0.1+0.01+0.001+ \dots \text{ upto } n \text{ terms}] \\
 & = 3 \left[ \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots + \frac{1}{10^n} \right] \\
 & = 3 \times a \left( \frac{1-r^n}{1-r} \right) \text{ where } a = \frac{1}{10} \text{ and } r = \frac{1}{10} \\
 & = 3 \times \frac{1}{10} \left( \frac{1-\frac{1}{10^n}}{1-\frac{1}{10}} \right) \\
 & = \frac{1}{3} (1 - 0.1^n)
 \end{aligned}$$

**Ex 11)** Find the  $n^{\text{th}}$  term of the sequence

$$0.4, 0.44, 0.444, \dots$$

**Solution :** Here  $t_1=0.4$

$$\begin{aligned}
 t_2 & = 0.44 = 0.4 + 0.04 \\
 t_3 & = 0.444 = 0.4 + 0.04 + 0.004 \\
 & \dots \dots \dots \\
 t_n & = 0.4 + 0.04 + 0.004 + 0.0004 + \dots \\
 & \dots \dots \dots \text{ upto } n \text{ terms}
 \end{aligned}$$

here  $t_n$  is the sum of first  $n$  terms of a G.P.

with  $a = 0.4$  and  $r = 0.1$

$$t_n = 0.4 \left( \frac{1 - 0.1^n}{1 - 0.1} \right) = \frac{4}{9} [1 - (0.1)^n]$$

**Ex 12)** For a sequence, if  $S_n = 5(4^n - 1)$ , find the

$n^{\text{th}}$  term, hence verify that it is a G.P., Also find  $r$ .

**Solution :**  $S_n = 5(4^n - 1)$ ,  $S_{n-1} = 5(4^{n-1} - 1)$

$$\begin{aligned}
 \text{We know that } t_n & = S_n - S_{n-1} \\
 & = 5(4^n - 1) - 5(4^{n-1} - 1) \\
 & = 5(4^n) - 5 - 5(4^{n-1}) + 5 \\
 & = 5(4^n - 4^{n-1}) \\
 & = 5(4^n - 4^n \cdot 4^{-1}) \\
 & = 5(4^n) \left( 1 - \frac{1}{4} \right) \\
 & = 5(4^n) \left( \frac{3}{4} \right) \\
 \therefore t_{n+1} & = 5(4^{n+1}) \times \frac{3}{4}
 \end{aligned}$$

Consider  $\frac{t_{n+1}}{t_n} = \frac{5(4^{n+1})}{5(4^n)} = 4 = \text{constant}$ ,

$\forall n \in \mathbb{N}$ .

$\therefore r = 4$

$\therefore$  the sequence is a G.P.

**Ex 13)** Which term of the sequence

$$\sqrt{3}, 3, 3\sqrt{3}, \dots \text{ is } 243?$$

**Solution :** Here  $a = \sqrt{3}$ ,  $r = \sqrt{3}$ ,  $t_n = 243$

$$\therefore a \cdot r^{n-1} = 243$$

$$\sqrt{3} \cdot (\sqrt{3})^{n-1} = 243 = 3^5 = (\sqrt{3})^{10}$$

$$\therefore (\sqrt{3})^n = (\sqrt{3})^{10}$$

$$\therefore n = 10.$$

Tenth term of the sequence is 243.

**Ex 14)** How many terms of G.P.

$2, 2^2, 2^3, 2^4, \dots$  are needed to give the sum 2046.

**Solution :** Here  $a=2$ ,  $r = 2$ , let  $S_n = 2046$ .

$$\therefore 2046 = a \left( \frac{r^n - 1}{r - 1} \right) = 2 \left[ \frac{2^n - 1}{2 - 1} \right] = 2 (2^n - 1)$$

$$1023 = 2^n - 1, 2^n = 1024 = 2^{10} \therefore n = 10$$

**Ex 15)** Mr. Pritesh got the job with an annual salary package of Rs. 400000 with 10% annual increment. Find his salary in the 5th year and also find his total earnings through salary in 10 years.

[Given  $(1.1)^4 = 1.4641$ ,  $(1.1)^{10} = 2.59374$

**Solution :** In the first year he will get a salary of Rs. 400000 .

He gets an increment of 10% so in the second year his salary will be

$$400000 \times \left( \frac{110}{100} \right) = 440000$$

In the third year his salary will be

$$400000 \times \left( \frac{110}{100} \right)^2 \text{ and so on } \dots\dots\dots$$

Hence it is a G.P. with  $a = 400000$  &  $r = 1.1$ .

Similarly his salary in the fifth year will be

$$t_5 = ar^4 = 400000 \left( \frac{110}{100} \right)^4 = 585640.$$

$$[\because (1.1)^4 = 1.4641]$$

His total income through salary in 10 years

$$\text{will be } S_{10} = a \left( \frac{r^{10} - 1}{r - 1} \right)$$

$$= 400000 \times \left( \frac{2.59374 - 1}{0.1} \right)$$

$$[\because (1.1)^{10} = 2.59374]$$

$$= 400000 \left( \frac{1.59374}{0.1} \right)$$

$$= 400000 [ 15.9374 ] = 63,74,960.$$

$\therefore$  Mr. Pritesh will get Rs.5,85,640 in the fifth year and his total earnings through salary in 10 years will be Rs. 63,74,960.

**Ex 16)** A teacher wanted to reward a student by

giving some chocolates. He gave the student two choices. He could either have 50 chocolates at once or he could get 1 chocolate on the first day, 2 on the second day, 4 on the third day and so on for 6 days. Which option should the student choose to get more chocolates?

**Ans :** We need to find sum of chocolates in 6 days.

According to second option teacher gives 1 chocolate on the first day, 2 on the second day, 4 on the third day, and so on. Hence it is a G. P . with  $a = 1$ ,  $r = 2$ .

If the number of chocolates collected in this way is greater than 50 we have to assume this is the better way.

$$\begin{aligned} \text{By using } S_n &= a \left( \frac{r^n - 1}{r - 1} \right) \\ &= 1 \left( \frac{2^6 - 1}{2 - 1} \right) \\ &= 64 - 1 = 63 \end{aligned}$$

Hence the student should choose the second way to get more chocolates.

**EXERCISE 4.2**

- 1) For the following G.P.s, find  $S_n$ 
  - i) 3, 6, 12, 24, .....
  - ii)  $p, q, \frac{q^2}{p}, \frac{q^3}{p^2}, \dots\dots\dots$
- 2) For a G.P. if
  - i)  $a = 2, r = -\frac{2}{3}$ , find  $S_6$
  - ii)  $S_5 = 1023, r = 4$ , Find  $a$
- 3) For a G.P. if
  - i)  $a = 2, r = 3, S_n = 242$  find  $n$ .
  - ii) sum of first 3 terms is 125 and sum of next 3 terms is 27, find the value of  $r$ .
- 4) For a G.P.

- i) If  $t_3 = 20$ ,  $t_6 = 160$ , find  $S_7$   
 ii) If  $t_4 = 16$ ,  $t_9 = 512$ , find  $S_{10}$
- 5) Find the sum to  $n$  terms  
 i)  $3 + 33 + 333 + 3333 + \dots$   
 ii)  $8 + 88 + 888 + 8888 + \dots$
- 6) Find the sum to  $n$  terms  
 i)  $0.4 + 0.44 + 0.444 + \dots$   
 ii)  $0.7 + 0.77 + 0.777 + \dots$
- 7) Find the  $n^{\text{th}}$  term of the sequence  
 i)  $0.5, 0.55, 0.555, \dots$   
 ii)  $0.2, 0.22, 0.222, \dots$
- 8) For a sequence, if  $S_n = 2(3^n - 1)$ , find the  $n^{\text{th}}$  term, hence show that the sequence is a G.P.
- 9) If  $S, P, R$  are the sum, product and sum of the reciprocals of  $n$  terms of a G.P. respectively, then verify that  $\left(\frac{S}{R}\right)^n = P^2$ .
- 10) If  $S_n, S_{2n}, S_{3n}$  are the sum of  $n, 2n, 3n$  terms of a G.P. respectively, then verify that  $S_n(S_{3n} - S_{2n}) = (S_{2n} - S_n)^2$ .

#### 4.5 Sum of infinite terms of a G. P.

We have learnt how to find the sum of first  $n$  terms of a G.P.

If the G.P. is infinite, does it have a finite sum?

#### Let's understand

Let's find sum to infinity. How can we find it?

We know that for a G.P.

$$S_n = a \left( \frac{1-r^n}{1-r} \right) = \frac{a}{1-r} - \left( \frac{a}{1-r} \right) r^n$$

If  $|r| < 1$  then, as  $n$  tends to infinity,  $r^n$  tends to zero.

Hence  $S_n$  tends to  $\frac{a}{1-r}$ .

(as  $\left(\frac{a}{1-r}\right) r^n$  tends to zero)

Hence the sum of an infinite G.P. is given

by  $\frac{a}{1-r}$ , when  $|r| < 1$ .

**Note:** If  $|r| \geq 1$  then sum to infinite terms does not exist.

#### SOLVED EXAMPLES

**EX 1)** Determine whether the sum of all the terms in the series is finite?

In case it is finite find it.

i)  $\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \dots$

ii)  $1, \frac{-1}{2}, \frac{1}{4}, \frac{-1}{8}, \dots$

iii)  $-\frac{3}{5}, \frac{-9}{25}, \frac{-27}{125}, \frac{-81}{625}, \dots$

iv)  $1, -3, 9, -27, 81, \dots$

**Solution: i)**  $r = \frac{1}{3}$

Here  $a = \frac{1}{3}$ ,  $r = \frac{1}{3}$ ,  $|r| < 1$

$\therefore$  Sum to infinity exists.

$$S = \frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \left[ \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{3}\right)} \right] = \frac{1}{2}$$

ii) Here  $a = 1$ ,  $r = -\frac{1}{2}$ ,  $|r| < 1$

∴ Sum to infinity exist

$$S = \frac{a}{1-r} = \frac{1}{1 - (-\frac{1}{2})} = \frac{1}{\left(\frac{3}{2}\right)} = \frac{2}{3}.$$

iii) Here  $a = -\frac{3}{5}, r = \frac{3}{5}, |r| < 1$

∴ Sum to infinity exists

$$S = \frac{a}{1-r} = \frac{-\frac{3}{5}}{1 - \left(\frac{3}{5}\right)} = \frac{-\left(\frac{3}{5}\right)}{\left(\frac{2}{5}\right)} = \frac{-3}{2}.$$

iv) Here  $a=1, r = -3$

As  $|r| \not< 1$

∴ Sum to infinity does not exist.



**Let's learn.**

### RECURRING DECIMALS :

We know that every rational number has decimal form.

**For example ,**

$$\frac{7}{6} = 1.166666..... = 1.1\dot{6}$$

$$\frac{5}{6} = 0.833333..... = 0.8\dot{3}$$

$$\frac{-5}{3} = -1.666666 = -1.\dot{6}$$

$$\frac{22}{7} = 3.142857142857..... = 3.\overline{142857}$$

$$\frac{23}{99} = 0.23232323..... = 0.\overline{23}$$

We can use G.P. to represent recurring decimals as a rational number .

### SOLVED EXAMPLES

**Ex i)** 0.66666.....

$$= 0.6 + 0.06 + 0.006 + \dots\dots\dots$$

$$= \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \dots\dots\dots,$$

the terms are in G.P. with  $a = 0.6, r = 0.1 < 1$

∴ Sum to infinity exists and is given by

$$\frac{a}{1-r} = \frac{0.6}{1-0.1} = \frac{0.6}{0.9} = \frac{6}{9} = \frac{2}{3}$$

ii)  $0.\overline{46} = 0.46 + 0.0046 + 0.000046 + \dots\dots\dots$  the terms are in G.P. with  $a = 0.46, r = 0.01 < 1$ .

∴ Sum to infinity exists

$$= \frac{a}{1-r} = \frac{0.46}{1-(0.01)} = \frac{0.46}{0.99} = \frac{46}{99}.$$

iii)  $2.\overline{5} = 2 + 0.5 + 0.05 + 0.005 + 0.0005 + \dots\dots\dots$   
After the first term, the terms are in G.P. with  $a = 0.5, r = 0.1 < 1$

∴ Sum to infinity exists

$$= \frac{a}{1-r} = \frac{0.5}{1-0.1} = \frac{0.5}{0.9} = \frac{5}{9}$$

∴  $2.\overline{5} = 2 + 0.5 + 0.05 + 0.005 + 0.0005 + \dots\dots\dots$

$$= 2 + \frac{5}{9} = \frac{23}{9}$$

### EXERCISE 4.3

1) Determine whether the sums to infinity of the following G.P.s exist ,if exist find them

i)  $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\dots\dots$

ii)  $2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \dots\dots\dots$

iii)  $-3, 1, \frac{-1}{3}, \frac{1}{9}, \dots\dots\dots$

iv)  $\frac{1}{5}, \frac{-2}{5}, \frac{4}{5}, \frac{-8}{5}, \frac{16}{5}, \dots$

2) Express the following recurring decimals as a rational number.

i)  $0.\overline{32}$ .

ii)  $3.\overline{5}$

iii)  $4.\overline{18}$

iv)  $0.3\overline{45}$

v)  $3.4\overline{56}$

3) If the common ratio of a G.P. is  $\frac{2}{3}$  and sum of its terms to infinity is 12. Find the first term.

4) If the first term of a G.P. is 16 and sum of its terms to infinity is  $\frac{176}{5}$ , find the common ratio.

5) The sum of the terms of an infinite G.P. is 5 and the sum of the squares of those terms is 15. Find the G.P.



**Let's learn.**

### Harmonic Progression ( H. P. )

**Definition :** A sequence  $t_1, t_2, t_3, t_4, \dots, t_n$

(  $t_n \neq 0, n \in \mathbb{N}$  ) is called a harmonic progression if

$\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}, \dots, \frac{1}{t_n}, \dots$  are in A.P.

For example ,

i)  $\frac{1}{7}, \frac{1}{11}, \frac{1}{15}, \dots$

ii)  $\frac{1}{4}, \frac{1}{9}, \frac{1}{14}, \frac{1}{19}, \dots$

iii)  $\frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \frac{1}{14}, \dots$

iv)  $\frac{1}{4}, \frac{3}{14}, \frac{3}{16}, \frac{1}{6}, \dots$

### SOLVED EXAMPLES

**Ex1)** Find the  $n^{\text{th}}$  term of the H.P.

$\frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7}, \dots$

**Solution :** Here  $2, \frac{5}{2}, 3, \frac{7}{2}, \dots$  are

in A.P. with  $a = 2$  and  $d = \frac{1}{2}$  hence

$\frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7}, \dots$  are in H.P.

For A.P.

$t_n = a + (n-1)d = 2 + (n-1)\frac{1}{2}$

$= 2 + \frac{1}{2}n - \frac{1}{2}$

$t_n = \frac{3}{2} + \frac{n}{2} = \frac{3+n}{2}$

For H.P.  $t_n = \frac{2}{3+n}$

**Ex2)** Find the  $n^{\text{th}}$  term of H.P.  $\frac{1}{5}, 1, \frac{-1}{3}, \frac{-1}{7}, \dots$

**Solution:** Since  $5, 1, -3, -7, \dots$  are in A.P.

with  $a = 5$  and  $d = -4$

Hence  $t_n = a + (n-1)d$

$= 5 + (n-1)(-4)$

$= 5 - 4n + 4 = 9 - 4n.$

For H.P.  $t_n = \frac{1}{9-4n}$

### Activity : 4.9

Find the  $n^{\text{th}}$  term of the following H.P.

$\frac{1}{2}, \frac{1}{7}, \frac{1}{12}, \frac{1}{17}, \frac{1}{22}, \dots$

**Solution :** Here  $2, 7, 12, 17, 22, \dots$  are in

with  $a = \text{$  and  $(d) = \text{$

hence  $\frac{1}{2}, \frac{1}{7}, \frac{1}{12}, \frac{1}{17}, \frac{1}{22}, \dots$  are in

$$t_n = a + (n-1)d =$$

$$= \text{} + (n-1)\text{$$

$$\therefore t_n \text{ of the H.P.} = \text{$$



**Let's learn.**

**Types of Means:**

**Arithmetic mean (A. M.)**

If  $x$  and  $y$  are two numbers, their A.M. is defined

by  $A = \frac{x+y}{2}$  .

We observe that  $x, A, y$  form an AP.

**Geometric mean (G. M.)**

If  $x$  and  $y$  are two numbers having same sign (positive or negative), their G.M. is defined by

$G = \sqrt{xy}$  . We observe that  $x, G, y$  form a G.P. .

**Harmonic mean (H. M.)**

If  $x$  and  $y$  are two numbers, their H.M. is defined

by  $H = \frac{2xy}{x+y}$  .

We observe that  $x, H, y$  form an HP .

These definitions can be extended to  $n$  numbers as follows

$$A = \frac{x_1 + x_2 + x_3 + \dots + x_n}{n}$$

$$G = \sqrt[n]{x_1 \cdot x_2 \cdot x_3 \cdot \dots \cdot x_n}$$

$$H.M. = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}}$$

**Theorem :** If  $A, G$  and  $H$  are A.M., G.M., H.M. of two positive numbers respectively, then

i)  $G^2 = AH$  ii)  $H < G < A$

**Proof :** let  $x$  and  $y$  be the two positive numbers.

$A = \frac{x+y}{2}, G = \sqrt{xy}, H = \frac{2xy}{x+y}$

RHS =  $AH = \frac{x+y}{2} \cdot \frac{2xy}{x+y}$   
 $= xy = G^2 = \text{L.H.S.}$

Consider  $A-G = \frac{x+y}{2} - \sqrt{xy}$

$= \frac{1}{2}(x+y - 2\sqrt{xy})$

$A-G = \frac{1}{2}(\sqrt{x} - \sqrt{y})^2 > 0$

$\therefore A > G \dots \dots \dots \text{(I)}$

$\therefore \frac{A}{G} > 1 \dots \dots \dots \text{(II)}$

Now consider  $G^2 = AH$

$\frac{G}{H} = \frac{A}{G} > 1 \quad \text{(FROM II)}$

$\therefore \frac{G}{H} > 1 \therefore G > H \dots \dots \dots \text{(III)}$

From (I) and (III)  $H < G < A$

**Note :** If  $x = y$  then  $H = G = A$

**n arithmetic means between a and b :**

Let  $A_1, A_2, A_3, \dots$  be the  $n$  A.M.s between  $a$  and  $b$  ,

then  $a, A_1, A_2, A_3, \dots, A_n, b$  is an A.P.

Here total number of terms are  $n+2$

$b = t_{n+2} = a + [(n+2) - 1]d$

$b = a + (n+1)d$

$d = \frac{b-a}{n+1}$

$A_1 = a + d = a + \frac{b-a}{n+1}$

$A_2 = a + 2d = a + 2 \frac{b-a}{n+1}$

$$A_3 = a + 3d = a + 3 \frac{b-a}{n+1}$$

.

.

$$A_n = a + n d$$

$$= a + n \frac{b-a}{n+1} = \frac{a(n+1)}{n+1} + n \frac{b-a}{n+1}$$

$$= \frac{a(n+1) + n(b-a)}{n+1}$$

$$A_n = \frac{a+nb}{n+1} .$$

**n geometric means between a and b :**

Let  $G_1, G_2, G_3, G_4, \dots, G_n$  be the  $n$  G.M.s between  $a$  and  $b$ ,

then  $a, G_1, G_2, G_3, \dots, G_n, b$  are in G.P

Here total number of terms are  $n+2$

$$\therefore t_{n+2} = b = a(r)^{n+1}$$

$$\therefore r^{n+1} = \frac{b}{a}$$

$$\therefore r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$G_1 = ar = a \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$G_2 = ar^2 = a \left(\frac{b}{a}\right)^{\frac{2}{n+1}} \dots\dots\dots$$

$$G_n = ar^n = a \left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

**Examples based on means**

**Ex : 1** Find A.M.,G.M.,H.M. of the numbers 4 and 16

**Solution :** Let  $x = 4$  and  $y = 16$

$$A = \frac{x+y}{2} = \frac{20}{2} = 10 \therefore A = 10$$

$$G = \sqrt{xy} = \sqrt{64} = 8, G=8.,$$

$$H = \frac{2xy}{x+y} = \frac{2(4)(16)}{4+16} = \frac{128}{20} = \frac{32}{5} .$$

**Ex2)** Insert 4 arithmetic means between 2 and 22.

**Solution:**

let  $A_1, A_2, A_3, A_4$  be 4 arithmetic means between 2 and 22

$\therefore 2, A_1, A_2, A_3, A_4, 22$  are in AP with

$$a = 2, t_6 = 22, n = 6 .$$

$$\therefore 22 = 2 + (6-1)d = 2 + 5d$$

$$20 = 5d, d=4$$

$$A_1 = a+d = 2+4 = 6 ,$$

$$A_2 = a+2d = 2+2 \times 4 = 2+8 = 10,$$

$$A_3 = a+3d = 2+3 \times 4 = 2+12 = 14$$

$$A_4 = a+4d = 2 + 4 \times 4 = 2+16 = 18.$$

$\therefore$  the 4 arithmetic means between 2 and 22 are 6,10,14,18.

**Ex: 3** Insert two numbers between  $\frac{2}{9}$  and  $\frac{1}{12}$  so that the resulting sequence is a H.P.

**Solution :** let the required numbers be  $\frac{1}{H_1}$  and  $\frac{1}{H_2}$

$$\therefore \frac{2}{9}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{12} \text{ are in H.P.}$$

$\frac{9}{2}, H_1, H_2, 12$  are in A.P.

$$t_1 = a = \frac{9}{2}, t_4 = 12 = a+3d = \frac{9}{2} + 3d .$$

$$3d = 12 - \frac{9}{2} = \frac{24-9}{2} = \frac{15}{2}$$

$$d = \frac{5}{2}$$

$$t_2 = H_1 = a+d = \frac{9}{2} + \frac{5}{2} = \frac{14}{2} = 7.$$

$$t_3 = H_2 = a+2d = \frac{9}{2} + 2 \times \frac{5}{2} = \frac{19}{2} ..$$

For resulting sequence  $\frac{1}{7}$  and  $\frac{2}{19}$  are to be

inserted between  $\frac{2}{9}$  and  $\frac{1}{12}$

**Ex: 4** Insert two numbers between 1 and 27 so that the resulting sequence is a G. P.

**Solution:** Let the required numbers be  $G_1$  and  $G_2$

$\therefore 1, G_1, G_2, 27$  are in G.P.

$\therefore t_1=1, t_2=G_1, t_3=G_2, t_4=27$

$\therefore a=1, t_4=ar^3=27$

$\therefore r^3=27=3^3 \therefore r=3$

$t_2=G_1=ar=1 \times 3=3$

$t_3=G_2=ar^2=1(3)^2=9$

$\therefore 3$  and  $9$  are the two required numbers.

**Ex: 5** The A.M. of two numbers exceeds their G.M. by 2 and their H.M. by  $18/5$ . Find the numbers.

**Solution :** Given  $A=G+2 \therefore G=A-2$

Also  $A=H+\frac{18}{5} \therefore H=A-\frac{18}{5}$

We know that  $G^2=AH$

$(A-2)^2=A\left(A-\frac{18}{5}\right)$

$A^2-4A+2^2=A^2-\frac{18}{5}A$

$\frac{18}{5}A-4A=-4$

$-2A=-4 \times 5, \therefore A=10$

Also  $G=A-2=10-2=8$

$\therefore A=\frac{x+y}{2}=10, x+y=20, y=20-x$   
.....(i)

Now  $G=\sqrt{xy}=8 \therefore xy=64$

$\therefore x(20-x)=64$

$20x-x^2=64$

$x^2-20x+64=0$

$(x-16)(x-4)=0$

$x=16$  or  $x=4$ .

$\therefore$  If  $x=16$ , then  $y=4 \therefore y=20-x$

$\therefore$  If  $x=4$ , then  $y=16$ .

The required numbers are 4 and 16.

### EXERCISE 4.4

- Verify whether the following sequences are H.P.
  - $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots$
  - $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \dots$
  - $\frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \frac{1}{15}, \dots$
- Find the  $n^{\text{th}}$  term and hence find the 8<sup>th</sup> term of the following H.P.s
  - $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \dots$
  - $\frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \dots$
  - $\frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}, \dots$
- Find A.M. of two positive numbers whose G.M. and H. M. are 4 and  $\frac{16}{5}$
- Find H.M. of two positive numbers whose A.M. and G.M. are  $\frac{15}{2}$  and 6
- Find G.M. of two positive numbers whose A.M. and H.M. are 75 and 48
- Insert two numbers between  $\frac{1}{7}$  and  $\frac{1}{13}$  so that the resulting sequence is a H.P.
- Insert two numbers between 1 and  $-27$  so that the resulting sequence is a G.P.
- Find two numbers whose A.M. exceeds their

G.M. by  $\frac{1}{2}$  and their H.M. by  $\frac{25}{26}$

- 9) Find two numbers whose A.M. exceeds G.M. by 7 and their H.M. by  $\frac{63}{5}$ .



**Let's remember!**

- 1) For an A.P.  $t_n = a + (n-1)d$
- 2) For a G.P.  $t_n = ar^{n-1}$ .
- 3) A.M. of two numbers  $A = \frac{x+y}{2}$
- 4) G.M. of two numbers  $G = \sqrt{xy}$
- 5) H.M. of two numbers  $H = \frac{2xy}{x+y}$
- 6)  $G^2 = AH$
- 7) If  $x = y$  then  $A = G = H$  [where  $x$  and  $y$  are any two numbers]
- 8) If  $x \neq y$  then  $H < G < A$ .

**4.6 Special Series (sigma Notation)**

The symbol  $\sum$  (the Greek letter sigma) is used as the summation sign. The sum  $a_1 + a_2 + a_3 + a_4 + \dots + a_n$  is expressed as

$$\sum_{r=1}^n a_r \text{ (read as sigma } a_r, r \text{ going from 1 to } n)$$

**For example :**

$$\sum_{i=1}^n x_i = x_1 + x_2 + x_3 + \dots + x_n$$

$$\sum_{i=1}^9 x_i = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9$$

$$\sum_{i=3}^{10} x_i = x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10}$$

$$\sum_{i=1}^n x_i^2 = x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2$$

$$\sum_{i=1}^n x_i y_i = x_1 y_1 + x_2 y_2 + x_3 y_3 + \dots + x_n y_n$$

Let's write some important results using  $\sum$  notation

**Result: 1)**

The sum of the first  $n$  natural numbers

$$= \sum_{r=1}^n r = \frac{n(n+1)}{2}$$

**Result 2)**

The sum of squares of first  $n$  natural numbers

$$= \sum_{r=1}^n r^2 = \frac{n(n+1)(2n+1)}{6}$$

**Result 3)**

The sum of the cubes of the first  $n$  natural numbers

$$= \sum_{r=1}^n r^3 = \left(\frac{n(n+1)}{2}\right)^2$$

**4.6.1 Properties of Sigma Notation**

i)  $\sum_{r=1}^n k t_r = k \sum_{r=1}^n t_r,$

Where  $k$  is a non zero constant.

ii)  $\sum_{r=1}^n (a_r + b_r) = \sum_{r=1}^n a_r + \sum_{r=1}^n b_r$

iii)  $\sum_{r=1}^n 1 = n$

iv)  $\sum_{r=1}^n k = k \sum_{r=1}^n 1 = k n,$

Where  $k$  is a non zero constant.

**SOLVED EXAMPLES**

**Ex 1)** Evaluate  $\sum_{r=1}^n (8r - 7)$

**Solution :**

$$\begin{aligned} \sum_{r=1}^n (8r - 7) &= \sum_{r=1}^n 8r - \sum_{r=1}^n 7 \\ &= 8 \sum_{r=1}^n r - 7 \sum_{r=1}^n 1 = 8 \left(\frac{n(n+1)}{2}\right) - 7n \end{aligned}$$

$$= 4(n^2+n) - 7n = 4n^2 + 4n - 7n = 4n^2 - 3n.$$

**Ex 2)** Find  $\sum_{r=1}^{17} (3r - 5)$

**Solution :**  $\sum_{r=1}^{17} (3r - 5) = \sum_{r=1}^{17} 3r - \sum_{r=1}^{17} 5$

$$\begin{aligned} &= 3 \sum_{r=1}^{17} r - 5 \sum_{r=1}^{17} 1 \\ &= 3 \frac{17(17+1)}{2} - 5(17) \\ &= 3 \times 17 \times \frac{18}{2} - 85 \\ &= 3 \times 17 \times 9 - 85 \\ &= 51 \times 9 - 85 \\ &= 459 - 85 \\ &= 374. \end{aligned}$$

**Ex 3)** Find  $3^2 + 4^2 + 5^2 + \dots + 29^2$ .

**Solution:**  $3^2 + 4^2 + 5^2 + \dots + 29^2$

$$\begin{aligned} &= (1^2 + 2^2 + 3^2 + \dots + 29^2) - (1^2 + 2^2) \\ &= \sum_{r=1}^{29} r^2 - \sum_{r=1}^2 r^2 \\ &= \frac{29(29+1)(58+1)}{6} - \frac{2(2+1)(4+1)}{6} \\ &= 29 \times 30 \times \frac{59}{6} - 2 \times 3 \times \frac{5}{6} \\ &= 29 \times 5 \times 59 - 5 \\ &= 5(29 \times 59 - 1) = 5(1711 - 1) \\ &= 5(1710) = 8550 \end{aligned}$$

**Ex 4)** Find  $100^2 - 99^2 + 98^2 - 97^2 + \dots + 2^2 - 1^2$

**Solution :**  $100^2 - 99^2 + 98^2 - 97^2 + \dots + 2^2 - 1^2$

$$\begin{aligned} &= (100^2 + 98^2 + 96^2 + \dots + 2^2) - \\ &\quad (99^2 + 97^2 + 95^2 + \dots + 1^2) \\ &= \sum_{r=1}^{50} (2r)^2 - \sum_{r=1}^{50} (2r-1)^2 \end{aligned}$$

$$= \sum_{r=1}^{50} (4r^2 - 4r^2 + 4r - 1)$$

$$= \sum_{r=1}^{50} (4r - 1)$$

$$= \sum_{r=1}^{50} 4r - \sum_{r=1}^{50} 1$$

$$= 4 \sum_{r=1}^{50} r - \sum_{r=1}^{50} 1 = 4 \times \frac{50(50+1)}{2} - 50$$

$$= 2 \times 50 \times 51 - 50$$

$$= 50(2 \times 51 - 1)$$

$$= 50(101)$$

$$= 5050.$$

**Ex 5)** Find  $\sum_{r=1}^n \frac{1^2+2^2+3^2+\dots+r^2}{r+1}$

**Solution :**  $\sum_{r=1}^n \frac{1^2+2^2+3^2+\dots+r^2}{r+1}$

$$= \sum_{r=1}^n \frac{r(r+1)(2r+1)}{6(r+1)}$$

$$= \frac{1}{6} \sum_{r=1}^n (2r^2 + r)$$

$$= \frac{1}{6} \left( 2 \sum_{r=1}^n r^2 + \sum_{r=1}^n r \right)$$

$$= \frac{1}{6} \left[ 2 \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right]$$

$$= \frac{1}{6} \left[ \frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2} \right]$$

$$= \frac{n(n+1)(4n+5)}{36}.$$

**Ex 6)** Find  $1 \times 5 + 3 \times 7 + 5 \times 9 + 7 \times 11$

$\dots +$  upto  $n$  terms .

**Solution :** Consider first factor of each term.  $1, 3, 5, 7, \dots$  are in A.P. with  $a=1, d=2$ .

$$t_r = a + (r-1)d = 1 + (r-1)2 = 2r-1.$$

Also the second factors  $5, 7, 9, 11, \dots$  are in A.P. with  $a=5, d=2$ .

$$t_r = 5 + (r-1)2 = 5 + 2r - 2 = 2r + 3$$

$$\begin{aligned}
\therefore S_n &= \sum_{r=1}^n (2r-1)(2r+3) \\
&= \sum_{r=1}^n (4r^2 + 4r - 3) \\
&= 4 \sum_{r=1}^n r^2 + 4 \sum_{r=1}^n r - \sum_{r=1}^n 3 \\
&= 4 \frac{n(n+1)(2n+1)}{6} + 4 \frac{n(n+1)}{2} - 3n \\
&= n \left[ \frac{2(n+1)(2n+1)}{3} \right] + 2n(n+1) - 3n \\
&= \frac{n}{3} [2(2n^2 + n + 2n + 1) + 6(n+1) - 9] \\
&= \frac{n}{3} [(4n^2 + 6n + 2) + 6n - 3] \\
&= \frac{n}{3} (4n^2 + 12n - 1)
\end{aligned}$$

**Ex 7)** If  $\frac{2+4+6+\dots+\text{upto } n \text{ terms}}{1+3+5+\dots+\text{upto } n \text{ terms}} = \frac{20}{19}$ ,

Find the value of  $n$ .

**Solution :**

Here the terms in the numerator are even numbers hence the general term is  $2r$ , terms in the denominator are odd numbers hence the general term is  $2r-1$ .

$$\therefore \frac{2+4+6+\dots+\text{upto } n \text{ terms}}{1+3+5+\dots+\text{upto } n \text{ terms}} = \frac{20}{19}$$

$$\frac{\sum_{r=1}^n 2r}{\sum_{r=1}^n (2r-1)} = \frac{20}{19}$$

$$\frac{2 \sum_{r=1}^n r}{2 \sum_{r=1}^n r - \sum_{r=1}^n 1} = \frac{20}{19}$$

$$2 \frac{n(n+1)}{2} \times 19 = 20 \times 2 \frac{n(n+1)}{2} - 20 \times n$$

$$n(n+1) \cdot 19 = 20n(n+1) - 20n$$

dividing by  $n$  we get,  $19(n+1) = 20(n+1) - 20$

$$19n + 19 = 20n + 20 - 20$$

$$n = 19.$$

### EXERCISE 4.5

- Find the sum  $\sum_{r=1}^n (r+1)(2r-1)$
- Find  $\sum_{r=1}^n (3r^2 - 2r + 1)$
- Find  $\sum_{r=1}^n \frac{1+2+3+\dots+r}{r}$
- Find  $\sum_{r=1}^n \frac{1^3+2^3+\dots+r^3}{r(r+1)}$
- Find the sum  $5 \times 7 + 9 \times 11 + 13 \times 15 + \dots$  upto  $n$  terms.
- Find the sum  $2^2+4^2+6^2+8^2+\dots$  upto  $n$  terms
- Find  $(70^2 - 69^2) + (68^2 - 67^2) + (66^2 - 65^2) + \dots + (2^2 - 1^2)$
- Find the sum  $1 \times 3 \times 5 + 3 \times 5 \times 7 + 5 \times 7 \times 9 + \dots$   
 $(2n-1)(2n+1)(2n+3)$
- Find  $n$ , if 
$$\frac{1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \dots + \text{upto } n \text{ terms}}{1+2+3+4+\dots+\text{upto } n \text{ terms}} = \frac{100}{3}.$$
- If  $S_1, S_2$  and  $S_3$  are the sums of first  $n$  natural numbers, their squares and their cubes respectively then show that  $9S_2^2 = S_3(1+8S_1)$ .

### MISCELLANEOUS EXERCISE - 4

- In a G.P., the fourth term is 48 and the eighth term is 768. Find the tenth term.
- For a G.P.  $a = \frac{4}{3}$  and  $t_7 = \frac{243}{1024}$ , find the value of  $r$ .

- 3) For a sequence, if  $t_n = \frac{5^{n-2}}{7^{n-3}}$ , verify whether the sequence is a G.P. If it is a G.P., find its first term and the common ratio.
- 4) Find three numbers in G.P. such that their sum is 35 and their product is 1000.
- 5) Find 4 numbers in G.P. such that the sum of middle 2 numbers is  $\frac{10}{3}$  and their product is 1.
- 6) Find five numbers in G.P. such that their product is 243 and sum of second and fourth number is 10.
- 7) For a sequence  $S_n = 4(7^n - 1)$  verify whether the sequence is a G.P.
- 8) Find  $2 + 22 + 222 + 2222 + \dots$  upto  $n$  terms.
- 9) Find the  $n^{\text{th}}$  term of the sequence  $0.6, 0.66, 0.666, 0.6666, \dots$
- 10) Find  $\sum_{r=1}^n (5r^2 + 4r - 3)$
- 11) Find  $\sum_{r=1}^n r(r-3)(r-2)$
- 12) Find  $\sum_{r=1}^n \frac{1^2 + 2^2 + 3^2 + \dots + r^2}{2r+1}$
- 13) Find  $\sum_{r=1}^n \frac{1^3 + 2^3 + 3^3 + \dots + r^3}{(r+1)^2}$
- 14) Find  $2 \times 6 + 4 \times 9 + 6 \times 12 + \dots$  upto  $n$  terms.
- 15) Find  $12^2 + 13^2 + 14^2 + 15^2 + \dots + 20^2$
- 16) Find  $(50^2 - 49^2) + (48^2 - 47^2) + (46^2 - 45^2) + \dots + (2^2 - 1^2)$ .
- 17) In a G.P. if  $t_2 = 7, t_4 = 1575$  find  $r$
- 18) Find  $k$  so that  $k-1, k, k+2$  are consecutive terms of a G.P.
- 19) If  $p^{\text{th}}, q^{\text{th}}$  and  $r^{\text{th}}$  terms of a G.P. are  $x, y, z$  respectively, find the value of  $x^{q-r} \times y^{r-p} \times z^{p-q}$

